NON COMPACT BOUNDARIES OF COMPLEX ANALYTIC VARIETIES IN HILBERT SPACES

SAMUELE MONGODI AND ALBERTO SARACCO

ABSTRACT. We treat the boundary problem for complex varieties with isolated singularities, of complex dimension greater than or equal to 3, non necessarily compact, which are contained in strongly convex, open subsets of a complex Hilbert space H. We deal with the problem by cutting with a family of complex hyperplanes in the fashion of [2] and applying the first named author's result for the compact case [13].

1. INTRODUCTION

Let M be a smooth and oriented $(2m+1)$-dimensional real submanifold of some complex manifold X. A natural question arises, whether M is the boundary of an $(m+1)$-dimensional complex analytic subvariety of X. This problem, the so-called *boundary problem*, has been widely treated over the past fifty-five years when M is compact and X is $\mathbb{C}^n$ or $\mathbb{CP}^n$. For a review of the boundary problem see [15], chapter 6.

The case when M is a compact, connected curve in $X = \mathbb{C}^n$ ($m = 0$), has been first solved by Wermer [17] in 1958. In 1966, Stolzenberg [16] proved the same result when M is a union of smooth curves. Later on, in 1975, Harvey and Lawson in [8] and [9] solved the boundary problem in $\mathbb{C}^n$ and then in $\mathbb{CP}^n \setminus \mathbb{CP}^r$, in terms of holomorphic chains, for any m. The boundary problem in $\mathbb{CP}^n$ was studied by Dolbeault and Henkin, in [5] for $m = 0$ and in [6] for any m. Moreover, in these two papers the boundary problem is dealt with also for closed submanifolds (with negligible singularities) contained in q-concave (i.e. union of $\mathbb{CP}^q$'s) open subsets of $\mathbb{CP}^n$. This allows M to be non compact. The results in [5] and [6] were extended by Dinh in [4].

The main theorem proved by Harvey and Lawson in [8] is that if $M \subset \mathbb{C}^n$ is compact and maximally complex then M is the boundary of a unique holomorphic chain of finite mass [8, Theorem 8.1]. Moreover, if M is contained in the boundary $b\Omega$ of a strictly pseudoconvex domain Ω, then M is the boundary of a complex analytic subvariety of Ω, with isolated singularities [10] (see also [7]).

In [2] Della Sala and the second named author generalized this last theorem to a non compact, connected, closed and maximally complex submanifold M (of real dimension at least 3, i.e. $m \geq 1$) of the connected boundary $b\Omega$ of an unbounded weakly pseudoconvex domain $\Omega \subset \mathbb{C}^n$. The extension result is obtained via a method of "cut-extend-and-paste".

In [13] the first named author established the Harvey-Lawson theorem for maximally complex manifolds of real dimension at least 3 ($m \geq 1$) contained in a complex Hilbert space, under the addition of a technical hypothesis.

The aim of this paper is to combine the techniques of these last papers, in order to generalize the extension result to a non necessarily bounded, connected, closed and maximally complex submanifold M ($\dim_{\mathbb{R}} M \geq 5$, i.e. $m \geq 2$) of the connected boundary $b\Omega$ of a strongly convex unbounded domain Ω of a complex Hilbert space H. The precise definitions will be given in the following section. The main theorem we establish is the following:

Theorem 1.1. *Let H be a complex Hilbert space, and $M \subset H$ such that*

(i) *M is a smooth maximally complex manifold of real dimension $2m + 1 \geq 5$ (complex dimension $m \geq 2$);*

(ii) *$M \subset b\Omega \subset H$, where Ω is a strongly convex domain;*

(iii) *there exists an orthogonal decomposition $H = \mathbb{C}^{m+1} \times H'$ such that the orthogonal projection $p : H \to \mathbb{C}^{m+1}$, when restricted to M, is a closed immersion with transverse self-intersections;*

(iv) *M is quasi-locally compact.*

Then there exists a unique analytic set of finite dimension T in Ω with isolated singularities, such that the boundary of T is M.

The strategy behind the proof of Theorem 1.1 is similar to that used in [2] and it is actually a simplification of that one. First we get a local and semi-global extension (see section 3), through an Lewy-type extension theorem for Hilbert-valued CR-functions. Then we cut Ω with parallel complex-hyperplanes.

Hypotheses (ii) and (iv) are technicalities needed in order to assure that the slices of M are compact, so that we can apply the extension result in [13] (the slices of M are maximally complex, and property (iii) is inherited by the hyperplane). The high dimension of M is needed in order to get the maximal complexity of slices (in an Hilbert space a moments condition makes less sense than in $\mathbb{C}^n$). We give a simple example (see example 4.1) showing that relaxing hypothesis (ii) can lead to the slices of Ω (thus of M) being unbounded. On the other hand, hypothesis (iv) is unnecessary (because always satisfied) if the following topological conjecture (by Williamson and Janos, 1987 [18]) is true.

Conjecture 1.1. *A complete admissible metric* d *for a* σ*-compact, locally compact space* X *is always a Heine-Borel metric if*

$$Cl\{x \in X \mid d(x, x_0) < r\} = \{x \in X \mid d(x, x_0) \leq r\}, \ \forall x_0 \in X, \ \forall r > 0.$$

Using the conjecture of Williamson and Janos, we can get rid of one annoying technical hypothesis:

Theorem 1.2. *Assume Conjecture 1.1 is true.*
   Let H *be a complex Hilbert space, and* $M \subset H$ *such that*
   (i) *M is a smooth maximally complex manifold of real dimension $2m + 1 \geq 5$ (complex dimension $m \geq 2$);*
   (ii) *$M \subset b\Omega \subset H$, where Ω is a strongly convex domain;*
   (iii) *there exists an orthogonal decomposition $H = \mathbb{C}^{m+1} \times H'$ such that orthogonal the projection $p : H \to \mathbb{C}^{m+1}$, when restricted to M, is a closed immersion with transverse self-intersections.*

Then there exists a unique analytic set of finite dimension T in Ω with isolated singularities, such that the boundary of T is M.

2. Notations and definitions

In the following, we denote by H a complex Hilbert space and by $B(x, \rho)$ the (open) ball of center $x \in H$ and radius $\rho > 0$. We introduce the following *quasi-local* property (following the terminology of [12]).

Definition 2.1. We say that $K \subset H$ is *quasi-locally compact* if, for any $x \in H$, for any $\rho > 0$, the set $B(x, \rho) \cap K$ is relatively compact in H.

Definition 2.2. Given an open set $\Omega \subset H$ with smooth boundary, we call it *strongly convex at* $x \in b\Omega$ if the Hessian form of the boundary at x is positive defined, with all eigenvalues greater than a fixed $\varepsilon > 0$.
   We call $\Omega \subset H$ *strongly convex* if it is strongly convex at all its boundary points.

We recall one of the equivalent definitions of finite-dimensional analytic subvariety of an infinite dimensional complex space: $A \subset H$ is said to be a *finite-dimensional analytic subvariety* if for any $x \in H$ there exist an open neighborhood U of x and a finite-dimensional complex manifold W of U such that $A \cap U \subset W$ and $A \cap U$ is an analytic subvariety of W. See [13] for some examples and a discussion of the relations between this definitions and the others that can be found in the literature.

In this paper, M will denote a smooth finite-dimensional manifold in H, of real dimension $2m + 1$ greater than or equal to 5 and p will always be the projection whose existence is required in the third hypothesis of Theorem 1.1. $H_x(M)$ will be the holomorphic tangent to M at x, i.e. $H_x(M) = T_x M \cap JT_x M$, where J is the natural complex structure on $T_x H \cong H$.

M will also be required to be *maximally complex*, i.e. such that $\dim_{\mathbb{C}} H_x(M) = m$ at all points $x \in M$, since maximal complexity is a necessary condition for being the boundary of a complex variety.

Given a smooth real hypersurface S in H, we denote by $\mathcal{L}_x(S)$ the Levi form of S at the point x; we note that, if S is the boundary of a strongly convex open set Ω, then $\mathcal{L}_x(S)$ is positive definite for every $x \in S$, i.e. a strongly convex open set is strongly pseudoconvex.

3. The local and semi-global results

The aim of this section is to prove the local result. Let 0 be a point of $M \subset S$. We have the following inclusions of tangent spaces:

$$H \supset T_0(S) \supset H_0(S) \supset H_0(M);$$

$$T_0(S) \supset T_0(M) \supset H_0(M).$$

Lemma 3.1. *Let M be a maximally complex submanifold of a smooth real hypersurface $S \subset H$, $\dim_{\mathbb{R}}(M) = 2m + 1$, $m \geq 1$. Suppose that $\mathcal{L}_0(S)$ has all but at most m eigenvalues of the same sign. Then*

$$H_0(S) \not\supset T_0(M).$$

The proof follows the lines of that of the $\mathbb{C}^n$ case (proved in [2]). We reproduce it here for the reader's convenience.

Proof. Should the thesis fail, we would have the following chain of inclusions

$$H \supset T_0(S) \supset H_0(S) \supset T \supset T_0(M) \supset H_0(M),$$

where T is the smallest complex space containing $T_0(M)$ (since M is maximally complex, $\dim_{\mathbb{C}} T = m + 1$). Hence, we may choose, in a neighborhood of 0, local complex coordinates $z_k = x_k + iy_k$, $k = 1, \ldots, m+1$, $w_l = u_l + iv_l$, $l = m + 2, \ldots$, in such a way that:

- $H_0(M) = \mathsf{span}\,(\partial/\partial x_k, \partial/\partial y_k)$, $k = 1, \ldots, m$
- $T_0(M) = \mathsf{span}\,(\partial/\partial x_k, \partial/\partial y_k, \partial/\partial x_{m+1})$, $k = 1, \ldots, m$
- $T = \mathsf{span}\,(\partial/\partial x_k, \partial/\partial y_k)$, $k = 1, \ldots, m+1$
- $H_0(S) = \mathsf{span}\,(\partial/\partial x_k, \partial/\partial y_k, \partial/\partial u_l, \partial/\partial v_l)$, $k = 1, \ldots, m+1$, $l = m+2, \ldots, n-1$
- $T_0(S) = \mathsf{span}\,(\partial/\partial x_k, \partial/\partial y_k, \partial/\partial u_{m+2}, \partial/\partial u_l, \partial/\partial v_l)$, $k = 1, \ldots, m+1$, $l \geq m+3$

We denote by z the first $m+1$ coordinates, by $\hat{z}$ the first m, and by π the projection on T; π is obviously a local embedding of M near 0, and we set $M_0 = \pi(M)$.

Locally at 0, S is a graph over its tangent space:

$$S = \{v_{m+2} = h(u_{m+2}, u_j, v_j, x_i, y_i)\}.$$

Observe that the Levi form of h has all but at most m eigenvalues of the same sign. In order to obtain a similar description of M, we proceed as follows. First, we have

$$M_0 = \{(\hat{z}, z_{m+1}) : y_{m+1} = \varphi(\hat{z}, x_{m+1})\}.$$

Then, we choose $f_j(\hat{z}, x_{m+1}) = f_j^1(\hat{z}, x_{m+1}) + if_j^2(\hat{z}, x_{m+1})$ (where f_j^1 and f_j^2 are real-valued) defined in a neighborhood of M_0 in T in such a way that

$$M = \{w_{m+2} = f_{m+2}(\hat{z}, x_{m+1}), \ldots, w_n = f_n(\hat{z}, x_{m+1})\}.$$

Observe that the function $(f_{m+2}(\hat{z}, x_{m+1}), \ldots, f_n(\hat{z}, x_{m+1}))$ is just $\pi^{-1}|_{M_0}$, and since M is maximally complex it has to be a CR map.

By hypothesis, the following equation holds in a neighborhood of 0:

$$f_{m+2}^2(\hat{z}, x_{m+1}) = h\left(f_{m+2}^1(\hat{z}, x_{m+1}), f_j^k(\hat{z}, x_{m+1}), \hat{z}, x_{m+1}\right).$$

After a computation of the second derivatives, taking into account that all first derivatives of f_j^k, of h and of φ vanish in the origin, we obtain

$$\frac{\partial^2 f_{m+2}^2}{\partial z_j \partial \overline{z}_k}(0) = \frac{\partial^2 h}{\partial z_j \partial \overline{z}_k}(0),$$

i.e. the Levi form of h and f_{m+2}^2 coincide in $H_0(M)$. By hypothesis $\mathcal{L}_0(h)$ is strictly positive definite on a non-zero subspace of $H_0(M)$. We shall obtain a contradiction by showing that $\mathcal{L}_0(f_{m+2})$ (and hence $\mathcal{L}_0(f_{m+2}^2)$) vanishes on $H_0(M)$. Let $\xi \in H_0(M)$. We may assume (up to unitary linear transformation of coordinates of $H_0(M)$) that $\xi = \partial/\partial z_1$.

Set $f \doteq f_{m+2}$. Then, since f is a CR function on M_0, we have:

$$\frac{\partial}{\partial \overline{z}_k} f(\hat{z}, x_{m+1}) = -\alpha(\hat{z}, x_{m+1})\frac{\partial}{\partial \overline{z}_k}\varphi(\hat{z}, x_{m+1}), \quad k = 1, \ldots, m$$

and

$$\frac{\partial}{\partial \overline{z}_{m+1}} f(\hat{z}, x_{m+1}) = -i\alpha(\hat{z}, x_{m+1}) + \alpha(\hat{z}, x_{m+1})\frac{\partial}{\partial x_{m+1}}\varphi(\hat{z}, x_{m+1}),$$

where $\alpha(\hat{z}, x_{m+1})$ is a complex valued function. Differentiating and calculating in 0 we obtain

$$(1) \qquad\qquad \frac{\partial^2 f}{\partial z_1 \partial \overline{z_1}}(0) = \alpha(0)\frac{\partial^2 \varphi}{\partial z_1 \partial \overline{z_1}}(0),$$

$$(2) \qquad\qquad 0 = \frac{\partial f}{\partial x_{m+1}}(0) = i\alpha(0),$$

i.e. $\alpha(0) = 0$. From (1) we deduce that $\partial^2 f/\partial z_1 \partial \overline{z}_1(0) = 0$. Contradiction. $\qquad\square$

The following lemma is an immediate consequence of a well-known fact.

Lemma 3.2. *Let $\mathcal{A}(D)$ be a Banach algebra (over $\mathbb{K} = \mathbb{R}, \mathbb{C}$) of continuous functions on D. If all the functions in $\mathcal{A}$ extends univocally to functions in $\mathcal{B}(D')$ (i.e., $D' \supset D$; $\mathcal{A}(D) = \mathcal{B}(D')$), then*

$$\| \hat{f} \|_{\mathcal{B}(D')} = \| f \|_{\mathcal{A}(D)},$$

where $\hat{f}$ is the unique extension of f.

Proof. Let $f \in \mathcal{A}(D)$. Let x be any point in D'. We can define

$$\chi_x : A(D) \to \mathbb{K} \quad \chi_x(f) = \hat{f}(x),$$

where $\hat{f}$ is the unique extension of f. χ_x is a character of the Banach algebra $\mathcal{A}(D)$, therefore continuous of unitary norm. Thus

$$|\hat{f}(x)| = |\chi_x(f)| \leq \| f \|_{\mathcal{A}(D)}, \quad \forall x \in D'.$$

Hence the thesis. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 3.3. *In the setting of Lemma 3.1, assume that S is the boundary of an unbounded domain $\Omega \subset H$, $0 \in M$ and that the Levi form of S has at most m non-positive eigenvalues. Then*

- *(i) there exists an open neighborhood U of 0 and an $(m + 1)$-dimensional complex submanifold $W_0 \subset U$ with boundary, such that $bW_0 = M \cap U$;*
- *(ii) $W_0 \subset \Omega \cap U$.*

Proof. To prove the first assertion, observe that to obtain $\mathcal{L}_0^M(\zeta_0, \overline{\zeta}_0)$ it suffices to choose a smooth local section ζ of $H_0(M)$ such that $\zeta(0) = \zeta_0$ and compute the projection of the bracket $[\zeta, \overline{\zeta}](0)$ on the real part of $T_0(M)$. By hypothesis, the intersection of the space where $\mathcal{L}_0(S)$ is positive with $H_0(M)$ is non empty; take η_0 in this intersection. Then $\mathcal{L}_0^M(\eta_0, \overline{\eta}_0) \neq 0$. Suppose, by contradiction, that the bracket $[\eta, \overline{\eta}](0)$ lies in $H_0(M)$, i.e. its projection on the real part of the tangent of M is zero. Then, if $\widetilde{\eta}$ is a local smooth extension of the field η to S, we have $[\widetilde{\eta}, \overline{\widetilde{\eta}}](0) = [\eta, \overline{\eta}](0) \in H_0(M)$. Since $H_0(M) \subset H_0(S)$, this would mean that the Levi form of S in 0 is zero in η_0. Now, we project (generically) M over a $\mathbb{C}^{m+1}$ in such a way that the projection $\pi : H \to \mathbb{C}^{m+1}$ is a local embedding of M near 0: since the restriction of π to M is a CR function, and since the Levi form of M has - by the arguments stated above - at least one positive eigenvalue, it follows that the Levi form of $\pi(M)$ has at least one positive eigenvalue. Thus, in order to obtain W_0, it is sufficient to apply the Lewy extension theorem [11] to the CR function $\pi^{-1}|_{\pi(M)}$.

In order to ensure that the extension lies in the Hilbert space H, we consider the orthonormal decomposition found before $H = \mathbb{C}^{m+1} \times H'$. Let $\mathbf{e}_j$ be a complex base of H', and $\pi_j^{-1}(M)$ the $\mathbf{e}_j$ coordinate of $\pi^{-1}|_{\pi(M)}$. We can apply Lewy's theorem to extend all of the functions π_j^{-1} to a fixed one-sided open neighbourhood U of $0 \in \pi(M)$; let us provisionaly denote by p_j the extension of π_j^{-1}.

For any positive integer k, for any $k-$tuple $(i_1, \ldots, i_k)$ and for any $a \in \mathbb{C}^k$, the scalar function

$$f = a_1 \pi_{i_1}^{-1} + \ldots + a_k \pi_{i_k}^{-1}$$

extends to U by

$$F = a_1 p_{i_1} + \ldots + a_k p_{i_k} \, .$$

Therefore, by Lemma 3.2, we know that

$$\|F\|_U \leq \|f\|_{\pi(M)} \leq \|a\|_{\mathbb{C}^k} \left\| \left(\sum_{j=1}^k |\pi_{i_j}^{-1}|^2 \right)^{1/2} \right\|_{\pi(M)} .$$

Let $z_0 \in U$ and take $\overline{a}_j = p_{i_j}(z_0)$ for $j = 1, \ldots, k$. Then

$$|p_{i_1}(z_0)|^2 + \ldots + |p_{j_k}(z_0)|^2 \leq \|\overline{p_{i_1}(z_0)} p_{i_1}(z) + \ldots + \overline{p_{i_k}(z_0)} p_{i_k}(z)\|_U$$

$$\leq \left(|p_{i_1}(z_0)|^2 + \ldots + |p_{j_k}(z_0)|^2 \right)^{1/2} \left\| \left(\sum_{j=1}^k |\pi_{i_j}^{-1}|^2 \right)^{1/2} \right\|_{\pi(M)}$$

and, letting z_0 vary in U, we have

$$\left\| \left(\sum_{j=1}^k |p_{i_j}|^2 \right)^{1/2} \right\|_U \leq \left\| \left(\sum_{j=1}^k |\pi_{i_j}^{-1}|^2 \right)^{1/2} \right\|_{\pi(M)} .$$

This implies that, if the sequence of the partial sums of

$$\sum_i \mathbf{e_i} \pi_i^{-1}$$

is a Cauchy sequence on $\pi(M)$ with respect to the supremum norm, then the same holds true for the sequence of partial sums of

$$\sum_i \mathbf{e_i} p_i$$

on U with respect to the supremum norm, implying the convergence of the latter to a holomorphic map from U to H'.

As for the second statement, we observe that the projection by π of the normal vector of S pointing towards Ω lies into the domain of $\mathbb{C}^{m+1}$ where the above extension W_0 is defined. Indeed, the extension result in [11] gives a holomorphic function in the connected component of (a neighborhood of 0 in) $H \setminus \pi(M)$ for which $\mathcal{L}_0(\pi(M))$ has a positive eigenvalue, when $\pi(M)$ is oriented as the boundary of this component. This is precisely the component towards which the projection of the normal vector of S points, when the orientations of S and M are chosen accordingly. This fact, combined with Lemma 3.1 (which states that any extension of M must be transverse to S) implies that locally $W_0 \subset \Omega \cap U$. $\qquad \square$

Corollary 3.4 (Semi global existence). *Let M be a maximally complex submanifold of the smooth boundary S of an unbounded domain $\Omega \subset H$, $0 \in M$, $\dim_{\mathbb{R}}(M) = 2m + 1$, $m \geq 1$. Assume that*

    (1) *M is quasi-locally compact;*
    (2) *the Levi form of S has at most m non-positive eigenvalues, at each point of M.*

Then there exist an open tubular neighborhood I of $S = b\Omega$ in $\overline{\Omega}$ and an $(m+1)$-dimensional complex submanifold W_0 of $\overline{\Omega} \cap I$, with boundary, such that $S \cap bW_0 = M$.

Proof. By Lemma 3.3, for each point $x \in M$, there exist a neighborhood U_x of x and a complex manifold $W_x \subset \overline{\Omega} \cap U_x$ bounded by M. Since M is quasi-locally compact, we can cover M with countable many such open sets U_i, and consider the union $W_0 = \cup_i W_i$. W_0 is contained in the union of the U_i's, hence we may restrict it to a tubular neighborhood I_M of M. It is easy to extend I_M to a tubular neighborhood I of S.

The fact that $W_i|_{U_{ij}} = W_j|_{U_{ij}}$ if $U_i \cap U_j = U_{ij} \neq \emptyset$ immediately follows from the construction made in Lemma 3.3, in view of the uniqueness of the holomorphic extension of CR functions. $\square$

4. THE GLOBAL RESULT

In this section we will prove Theorem 1.1.

Since Ω is strongly convex, we can find a real hyperplane

$$I = \{\mathfrak{Re}\, z_0 = 0\}$$

tangent to $b\Omega$ in 0, such that, for every translation

$$I_a = \{\mathfrak{Re}\, z_0 = a\}, \quad a \in \mathbb{R}^+$$

of I, $I_a \cap \overline{\Omega}$ is bounded (and not empty) and the same holds for nearby hyperplanes. Denoting by

$$L_k = \{z_0 = k\}, \quad k \in \mathbb{C}, \mathfrak{Re}\, k \in \mathbb{R}^+,$$

also $A_k = L_k \cap M$ is bounded and, by the quasi-local compactness of M, the slice A_k of M is compact. In view of Sard's lemma, up to modifying the equation of L_k by

$$L_k = \{z_0 + \varepsilon z_1 = k\}$$

we can suppose the slice A_k to be smooth and a transversal intersection, hence of the correct dimension $(2m - 1)$. As a notation, we'll call suitable a slicing hyperplane L_k that leads to a smooth, compact, transversal intersection, as above.

Thanks to the maximal complexity of M, it follows that each slice A_k is maximally complex too. Moreover, the technical hypothesis (iii) of Theorem 1.1 is inherited by the slice.

Fix a point in Ω. To this correspond a suitable slicing hyperplane L_{k_0} of the above form, such that nearby parallel hyperplanes are suitable too. Each slice A_k, k in a neighborhood U of k_0, satisfies the hypotheses

of the theorem in [13], thus is the boundary of a holomorphic chain $\tilde{A}_k$ with support in the hyperplane L_k, which is a smooth manifold near $b\Omega$, since there it coincides with the manifold obtained in Corollary 3.4.

Our goal is now to glue together the slices $\tilde{A}_k$: $\tilde{A}_U = \cup_{k \in U} \tilde{A}_k$ and show that $\tilde{A}_U$ is a holomorphic chain, too (without singularities near $b\Omega$, due to Corollary 3.4).

It is worth observing that a strictly convexity hypothesis does not suffice to use our slicing method, as the following example shows.

Example 4.1. Let $\Omega \subset H$ be defined by

$$\Omega = \left\{ (z_n)_{n \in \mathbb{N}} \in H \mid x_0 > y_0^2 + \sum_{n=1}^{\infty} \frac{|z_n|^2}{n} \right\}.$$

Then Ω is strictly convex (i.e. convex and its boundary does not contain lines or line segments). But it is not strongly convex at $0 \in b\Omega$.

Observe that the real tangent hyperplane

$$T_0 b\Omega = \{ (z_n)_{n \in \mathbb{N}} \in H \mid x_0 = 0 \}$$

is such that all its translated in the positive x_0 direction intersect Ω and $b\Omega$ in unbounded sets. That is also true for complex hyperplanes of the form

$$L_k = \{ (z_n)_{n \in \mathbb{N}} \in H \mid z_0 = k \}, \quad \mathfrak{Re}\, k > 0.$$

Hence it is not possible to apply out slicing method to a maximally complex manifold $M \subset b\Omega$, since we have no way to assure even the boundedness of the slice.

If we show that $\widetilde{A}_U = \bigcup_{k \in U} \widetilde{A}_k$ is an analytic space, the thesis will follow. By [13, Remark 5.4], $\widetilde{A}_U$ is a continuous family in the parameter k, therefore it is a rectifiable set of real dimension $2m + 2$. We denote by $[\widetilde{A}_U]$ the current of integration associated to it and we define the map $\kappa : \widetilde{A}_U \to U \subset \mathbb{C}$ such that $x \in \widetilde{A}_{\kappa(x)}$ for every $x \in \widetilde{A}_U$; the map κ is Lipschitz-continuous, therefore

$$\int_{\widetilde{A}_U} C(d^{\widetilde{A}_U} \kappa_x) \omega = \int_U \int_{\widetilde{A}_k} \omega$$

by the Coarea formula in [1, Theorem 9.4].

The previous formula implies that the current $[\widetilde{A}_U]$ is of bidimension $(m+1, m+1)$, which is equivalent to the fact that $\mathrm{Tan}^{(2m+2)}(\widetilde{A}_U, x) = V_x$ is a complex subspace for $\mathcal{H}^{2m+2}-$a.e. $x \in \widetilde{A}_U$; moreover, κ is the restriction to $\widetilde{A}_U$ of a $\mathbb{C}$-linear map $f : H \to \mathbb{C}^2$, therefore $d^{\widetilde{A}_U} \kappa_x = df|_{V_x}$. By formula (9.2) in [1] and the properties of $\mathbb{C}-$linear maps, we get

$$C(d^{\widetilde{A}_U} \kappa_x) = C(df|_{V_x}) > 0.$$

This implies that $[\widetilde{A}_U]$ is a positive current.

The topological boundary of $\widetilde{A}_U$ is given by the union

$$\bigcup_{k \in U} A_k \cup \bigcup_{k \in bU} \widetilde{A}_k$$

and therefore is again a rectifiable set, this time of dimension $2m + 1$. The boundary of the current $[\widetilde{A}_U]$ is concentrated on such a set and is therefore not contained in the bounded open set

$$\Omega_U = \bigcup_{k \in U} (L_k \cap \Omega) \ .$$

Summing up, $[\widetilde{A}_U]$ is a $(2m + 2)-$rectifiable current, which is positive and closed in Ω_U; therefore, by [13, Theorem 4.5], $[\widetilde{A}_U]$ can be represented by integration on the regular part of an analytic set. We denote such a set by V.

Let us consider the projection $p : H \to \mathbb{C}^{m+1}$, which is an immersion with self-transverse intersections when restricted to M, and let us suppose that, for some open set Ω_U, we can find a linear functional $\nu : \pi(\Omega_U) \to \mathbb{C}$ such that $p(A_k) = \nu^{-1}(k) \cap p(M)$ for every $k \in U$ and such that $p|_{A_k}$ is again an immersion with self-transverse intersections.

We can always find such a ν, up to shrinking U; we can also restrict U further, so that every connected component U_j of $\pi(\Omega_U) \setminus p(M)$ intersects $\nu^{-1}(k)$ in a non empty set for every $k \in U$. Going through the proof of Theorem 5.6 in [13], we can construct holomorphic functions

$$F_{j,k}^h : U_j \cap \nu^{-1}(k) \to H'$$

which realize $\widetilde{A}_k$ as their graph. What we proved before is that

$$U_j \ni (z, k) \mapsto F_{j,k}^h(z) = F_j^h(z, k)$$

is a holomorphic function whenever $(z, F_{j,k}^h(z))$ belongs to $(\widetilde{A}_k)_{\mathrm{reg}}$; therefore, we have an analytic set $S_{j,k} \subset U_j \cap \nu^{-1}(k)$ outside which the dependence from k is analytic. Let us denote by $S_j = \bigcup_k S_{j,k}$.

By an easy coarea argument, we observe that $\mathcal{H}^{2m+1}(S_j) = 0$.

Finally, the functions F_j^h are bounded on U_j because their images are contained in any ball which contains $\Omega_U \cap M$, which is bounded. Therefore, we can extend the functions F_j^h as holomorphic functions through S_j. Obviously, the graph of F_j^h on U_j coincides with the closure of its graph on $U_j \setminus S_j$; therefore, the collection of the graphs of the $F_j's$ (which are finite-dimensional analytic subspaces of $H \setminus M$ by Théorème 2 in the third part of [14]) supports the current $[\widetilde{A}_U]$, which is then a holomorphic chain.

The open sets

$$\omega_U \ \dot{=} \ \bigcup_{k \in U} L_k$$

are a covering of M. Since M is quasi-locally compact, we can find a locally finite countable subcovering ω_i, $i \in \mathbb{N}$. In each ω_i lives a

holomorphic chain $\tilde{A}_i$. On the intersection of two such open domains ω_i and ω_j the two chains coincide by analiticity, since they both coincide near the boundary $b\Omega$ with the manifold W of Corollary 3.4.

The union $T = \tilde{A}_i$ of the holomorphic chains defined in the open sets ω_i is the holomorphic chain with boundary M.

A finite dimensional analytic variety in H is contained in a finite dimensional complex manifold and it is a complex variety in the latter. Therefore, we can repeat almost verbatim the argument used in [2] to show that the singularities of $\widetilde{A}_U$ are a discrete set.

□

We remark that the previous proof works also in the finite-dimensional case, giving a simplification of the argument used in [2], by employing the classical result by King, instead of its Hilbert space analogue.

It is also worth noticing that we can to some extent relax the convexity property, asking only for Ω to be convex, strongly convex at one point and strongly pseudoconvex (or at least the Levi-form of $b\Omega$ to have at most m vanishing eigenvalues). In fact, strong convexity at one point and convexity everywhere ensure that the slices are compact, if the hyperplanes are parallel to the tangent at the point of strong convexity; we also need the strong pseudoconvexity assumption to guarantee that the Levi form of the boundary has positive eigenvalues, a fact which is implied by strong convexity but it isn't by mere convexity.

Proof of Theorem 1.2. It is sufficient to show that if Conjecture 1.1 holds true, then hypothesis (iv) is always satisfied.

We thus assume that Conjecture 1.1 holds true and we consider the metric space given by M with the distance d given by the restriction of the distance of H. Endowed with such distance, M is a locally compact space; as a manifold, M is second countable, therefore it is $\sigma-$compact.

As the closure of $\{\|x - x_0\| < \rho\}$ in H is $\{\|x - x_0\| \leq \rho\}$ for every $x_0 \in H$ and $\rho > 0$, the same holds when the open and closed balls are intersected with M. Therefore, the metric d is Heine-Borel, i.e. bounded closed sets are compact.

Now, let us take $x \in M$ and $\rho > 0$; by the Heine-Borel property, the set

$$\{y \in M \mid d(x, y) \leq \rho\} = M \cap \{y \in H \mid \|x - y\| \leq \rho\}$$

is compact, that is, M is quasi-locally compact, therefore we can apply Theorem 1.1 and obtain the desired result.   □

5. Further questions

It would be nice to get rid of the technical hypotheses (ii), (iii), (iv) or to find examples showing that the extension does not hold without them.

Hypothesis (ii) (or its weaker version explained after the proof of Theorem 1.1) is needed to apply the cut, extend and paste method as we presented it (see example 4.1), but might not be necessary to extension, as another line of proof might be possible.

Hypothesis (iii) is already present in the compact case treated in [13], and already in that case it would be nice to see whether it is a necessary request or not.

It is worth noticing that an example showing extension does not hold just under hypotheses (i), (ii) and (iii) would be an indirect proof of the falseness of Williamson and Janos' conjecture.

A possible direction for future research on the subject is that pursued in [3] in $\mathbb{C}^n$: given a (pseudo)convex domain $\Omega \subset H$, and a subdomain $A \subset b\Omega$ is it possible to find a domain $E \subset \Omega$ depending only on Ω and A such that every maximally complex manifold (of real dimension at least 5, satisfying some technical conditions) $M \subset A$ is the intersection of the boundary of a complex variety $W \subset E$ with A?

Thanks to what we proved in this paper, if Ω is a strongly convex domain, and $A = b\Omega$, then $E = \Omega$. Thus the question we are asking is indeed a generalization of the main result of this paper.

Another quite natural question is whether it is possible to extend this result (or the one contained in [13]) to Banach spaces.

References

[1] L. Ambrosio, B. Kirchheim, *Rectifiable sets in metric and Banach spaces*, Math. Ann., **318** 3 (2000), 527–555.

[2] G. Della Sala, A. Saracco, *Non-compact boundaries of complex analytic varieties*, Int. J. Math. **18** 2 (2007), 203–218.

[3] G. Della Sala, A. Saracco, *Semi-global extension of maximally complex submanifolds*, Bull. Aust. Math. Soc. **84** (2011), 458–474.

[4] T.-C. Dinh, *Conjecture de Globevnik-Stout et théorème de Morera pur une chaîne holomorphe*, Ann. Fac. Sci. Toulouse Math. **8** (1999) 235–257.

[5] P. Dolbeault and G. Henkin, *Surfaces de Riemann de bord donné dans $\mathbb{C}\mathbb{P}^n$*, in *Contributions to complex analysis and analytic geometry*, Aspects Math. (Vieweg, Braunschweig, 1994), pp. 163–187.

[6] P. Dolbeault and G. Henkin, *Chaînes holomorphes de bord donné dans $\mathbb{C}\mathbb{P}^n$*, Bull. Soc. Math. France **125** (1997) 383–445.

[7] M. P. Gambaryan, *Regularity condition for complex films*, Uspekhi Mat. Nauk **40** (1985) 203–204.

[8] F. Reese Harvey and H. Blaine Lawson Jr., *On boundaries of complex analytic varieties. I*, Ann. of Math. (2) **102** (1975), 223–290.

[9] F. R. Harvey and H. B. Lawson, Jr., *On boundaries of complex analytic varieties. II*, Ann. of Math. **106** (1977) 213–238.

[10] F. R. Harvey and H. B. Lawson, Jr., *Addendum to Theorem 10.4 in "Boundaries of analytic varieties"*, [arXiv: math.CV/0002195] (2000).

[11] H. Lewy, *On the local character of the solutions of an atypical linear differential equation in three variables and a related theorem for regular functions of two complex variables*, Ann. of Math. **64** (1956) 514–522.

[12] S. Mongodi, *Some applications of metric currents to complex analysis*, Man. Math. (to appear, 2012)

[13] S. Mongodi, *Positive metric currents and holomorphic chains in Hilbert spaces*, [arXiv:1207.5244] (2012).

[14] G. Ruget, *A propos des cycles analytiques de dimension infinie*, Inv. Math. **8** (1969) 267–312.

[15] A. Saracco, *Extension problems in complex and CR-geometry*, Tesi **9**, Edizioni della Normale, Pisa (2008), 150+xiv.

[16] G. Stolzenberg, *Uniform approximation on smooth curves*, Acta Math. **115** (1966) 185–198.

[17] J. Wermer, *The hull of a curve in $\mathbb{C}^n$*, Ann. of Math. **68** (1958) 550–561.

[18] R. Williamson, L. Janos, *Constructing metrics with the Heine-Borel property*, Proc. A.M.S. **100** 3 (1987), 567–573.

Samuele Mongodi, Scuola Normale Superiore, Piazza dei Cavalieri 7, I-56123 Pisa, Italy
E-mail address: s.mongodi@sns.it

Alberto Saracco, Dipartimento di Matematica e Informatica, Università di Parma, Parco Area delle Scienze 53/A, I-43124 Parma, Italy
E-mail address: alberto.saracco@unipr.it